# Finite element method (FEM1)

Lecture 4A. 8-node quadrilateral element

03.2026

8-node 2D quadrilateral element (accuracy, irregular shapes)

An 8-node element is defined by eight nodes with two degrees of freedom at each node: u_i, v_i. It provides more accurate results and can accommodate irregular shapes without significant loss of accuracy.

natural coordinate system
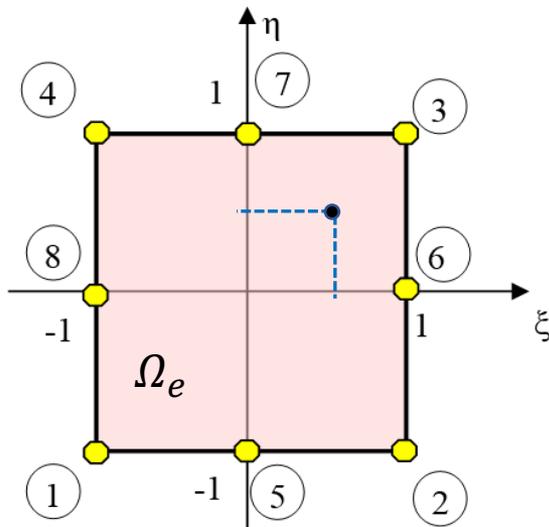
cartesian coordinate system

$n = 8$; $n_p = 2$ → $n_e = n \cdot n_p = 16$
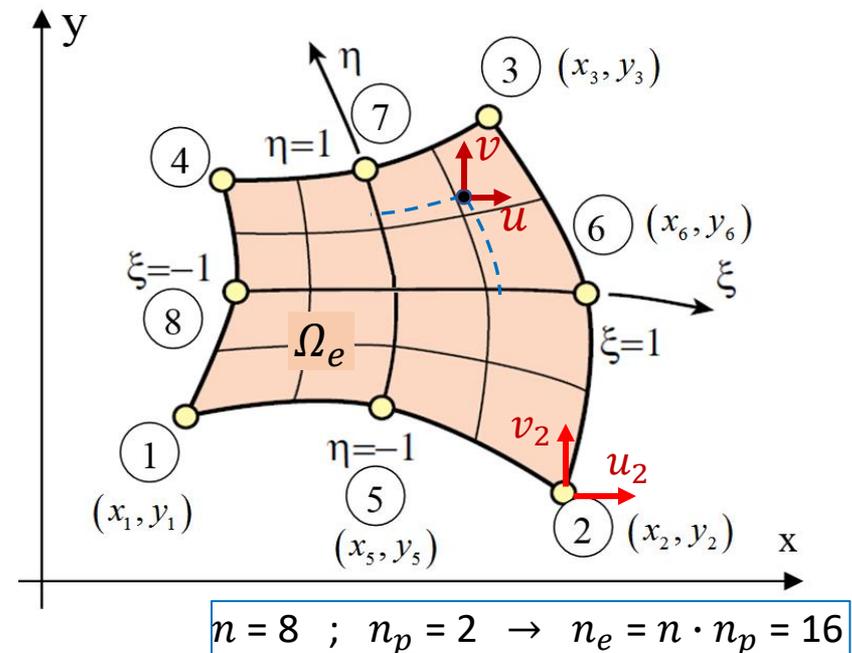
Geometry mapping: $(\xi, \eta) \to (x, y)$

$(-1,-1) \to (x_1, y_1)$ $(1,-1) \to (x_2, y_2)$ $(1,1) \to (x_3, y_3)$ $(-1,1) \to (x_4, y_4)$

$(0,-1) \to (x_5, y_5)$ $(1,0) \to (x_6, y_6)$ $(0,1) \to (x_7, y_7)$ $(-1,0) \to (x_8, y_8)$

2

Reference element vs real elements

One reference element maps to each real element of the mesh

reference element (parent element)
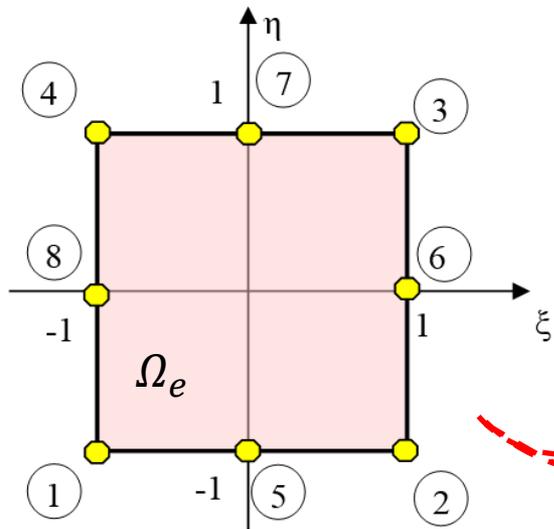
mesh of model elements

Isoparametric mapping
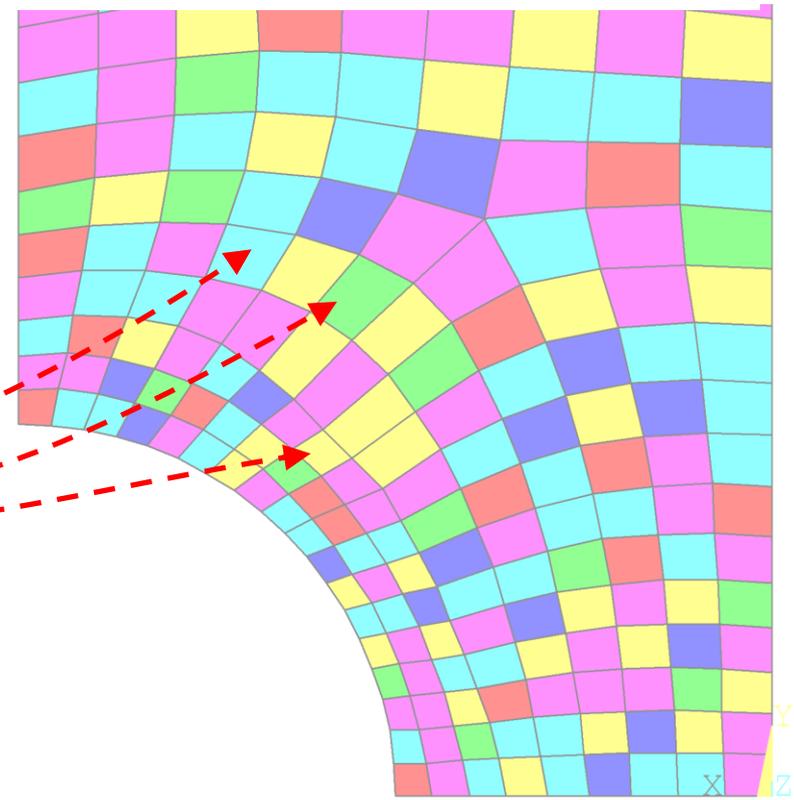
vectors of nodal coordinates

$$\{x_i\}_e = \begin{Bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ . \\ x_8 \end{Bmatrix}_{8 \times 1} \quad ; \quad \{y_i\}_e = \begin{Bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ . \\ y_8 \end{Bmatrix}_{8 \times 1}$$

local vector of nodal parameters
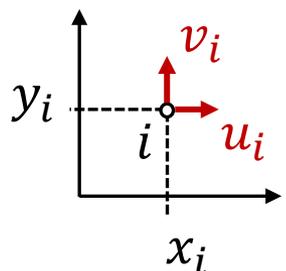
$$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ q_{16} \end{Bmatrix}_{e\ 16 \times 1} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ . \\ . \\ . \\ u_8 \\ v_8 \end{Bmatrix}_e$$

$$n = 8 \ ; \ n_p = 2 \quad \rightarrow \quad n_e = n \cdot n_p = 16$$
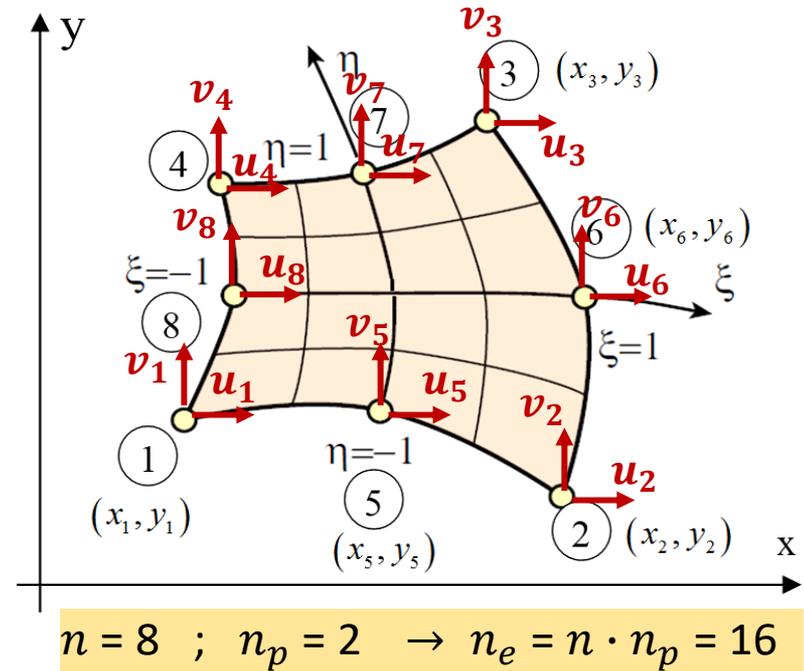
Isoparametric mapping – the same functions are used to describe the geometry and displacement field

Isoparametric mapping

matrix of shape functions*:*

$$[N(\xi,\eta)]_{2\times16} = \begin{bmatrix} N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & 0 & \dots & N_8(\xi,\eta) & 0 \\ 0 & N_1(\xi,\eta) & 0 & N_2(\xi,\eta) & \dots & 0 & N_8(\xi,\eta) \end{bmatrix}$$

vector of shape functions*:*

$$\lfloor N(\xi,\eta)\rfloor_{1\times8} = \lfloor N_1(\xi,\eta), N_2(\xi,\eta), \dots, N_8(\xi,\eta)\rfloor$$

position and displacement of any point:

$$x = \lfloor N(\xi,\eta)\rfloor_{1\times8} \{x_i\}_{e\ 8\times1} \quad ; \quad y = \lfloor N(\xi,\eta)\rfloor_{1\times8} \{y_i\}_{e\ 8\times1}$$

$$\{u\}_{2\times1} = \begin{Bmatrix} u \\ v \end{Bmatrix} = [N(\xi,\eta)]_{2\times16} \{q\}_{e\ 16\times1}$$
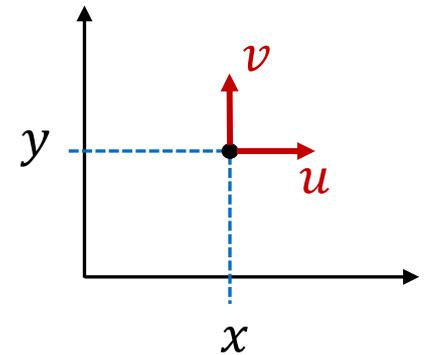
Shape functions of the 8-node quadrilateral finite element

corner nodes:

$N_2(\xi,\eta)$
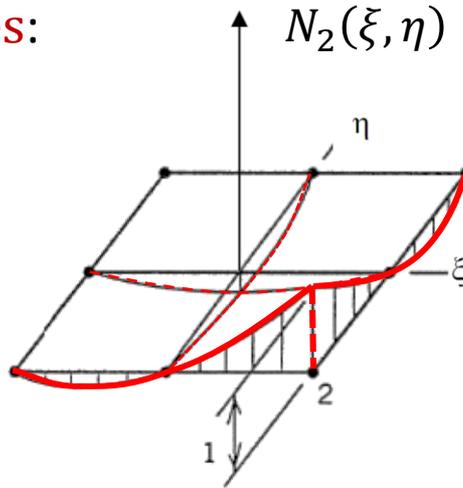
$$N_1(\xi,\eta) = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$N_2(\xi,\eta) = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

$$N_3(\xi,\eta) = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta)$$

$$N_4(\xi,\eta) = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$

midside nodes:

$N_6(\xi,\eta)$
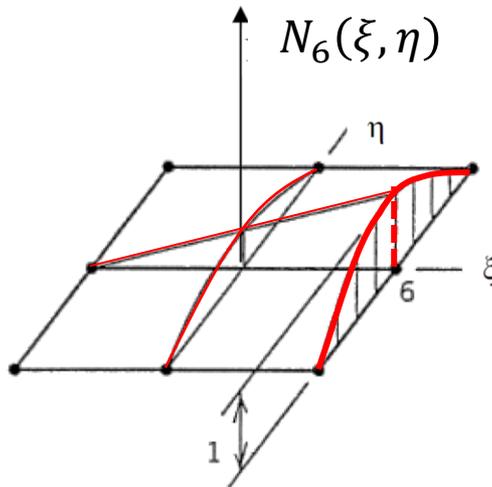
$$N_5(\xi,\eta) = \frac{1}{2}\left(1-\xi^2\right)(1-\eta)$$

$$N_6(\xi,\eta) = \frac{1}{2}(1+\xi)\left(1-\eta^2\right)$$

$$N_7(\xi,\eta) = \frac{1}{2}\left(1-\xi^2\right)(1+\eta)$$

$$N_8(\xi,\eta) = \frac{1}{2}(1-\xi)\left(1-\eta^2\right)$$

6

Transformation between natural and cartesian coordinates

partial derivatives of any function of coordinates (x, y) with respect to (ξ, η):

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\Longrightarrow \quad \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = [J]_{2\times2} \cdot \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix}$$

Jacobian matrix

partial derivatives of any function of coordinates (ξ, η) with respect to (x, y):

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\Longrightarrow \quad \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = [J]^{-1}_{2\times2} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix}$$

inverse Jacobian matrix

Transformation between natural and cartesian coordinates

differential operators:

$$\left\{\begin{matrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{matrix}\right\} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \left\{\begin{matrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{matrix}\right\} = [J]_{2\times 2} \cdot \left\{\begin{matrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{matrix}\right\} \qquad ; \qquad \left\{\begin{matrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{matrix}\right\} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \left\{\begin{matrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{matrix}\right\} = [J]^{-1}_{2\times 2} \left\{\begin{matrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{matrix}\right\}$$

Jacobian matrix   *inverse Jacobian matrix*

differential operators:

$$\left\{\begin{matrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{matrix}\right\} = [J]_{2\times 2} \left\{\begin{matrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{matrix}\right\} = [J]_{2\times 2} \cdot [J]^{-1}_{2\times 2} \left\{\begin{matrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{matrix}\right\} = [I]_{2\times 2} \cdot \left\{\begin{matrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{matrix}\right\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left\{\begin{matrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{matrix}\right\} = \left\{\begin{matrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{matrix}\right\}$$

unit matrix

How to find the inverse of the Jakobian matrix?

$$[J]=\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\[2mm] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

the inverse of the Jakobian matrix:

$$[J]^{-1}_{2\times2} = \frac{1}{det[J]}\left([J]^C\right)^T_{2\times2} = \frac{1}{det[J]}\left(\begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial x}{\partial \eta} \\[2mm] -\dfrac{\partial y}{\partial \xi} & \dfrac{\partial x}{\partial \xi} \end{bmatrix}\right)^T = \frac{1}{det[J]}\begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\[2mm] -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} =$$

$$=\begin{bmatrix} \dfrac{1}{det[J]}\dfrac{\partial y}{\partial \eta} & -\dfrac{1}{det[J]}\dfrac{\partial y}{\partial \xi} \\[2mm] -\dfrac{1}{det[J]}\dfrac{\partial x}{\partial \eta} & \dfrac{1}{det[J]}\dfrac{\partial x}{\partial \xi} \end{bmatrix}$$

transposed J's cofactor matrix

geometry approximation   *nodal coordinates*

$$[J]^{-1}_{2\times2} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\[2mm] \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \rightarrow$$

$$\frac{\partial \xi}{\partial x} = \frac{1}{det[J]}\frac{\partial y}{\partial \eta} = \frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \eta} = \frac{1}{det[J]}\underset{1\times8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\underset{8\times1}{\{y_i\}_e}$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{det[J]}\frac{\partial y}{\partial \xi} = -\frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \xi} = -\frac{1}{det[J]}\underset{1\times8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\underset{8\times1}{\{y_i\}_e}$$

$$\frac{\partial \xi}{\partial y} = -\frac{1}{det[J]}\frac{\partial x}{\partial \eta} = -\frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \eta} = -\frac{1}{det[J]}\underset{1\times8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\underset{8\times1}{\{x_i\}_e}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{det[J]}\frac{\partial x}{\partial \xi} = \frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \xi} = \frac{1}{det[J]}\underset{1\times8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\underset{8\times1}{\{x_i\}_e}$$

9

How to find the inverse of the Jakobian matrix?

determinant of the Jakobian matrix:

geometry approximation

$$det[J] = det\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfra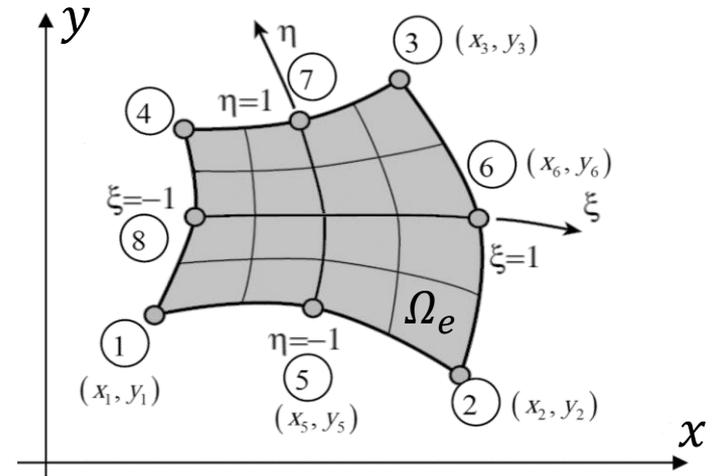c{\partial y}{\partial \eta} \end{bmatrix} = \dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \eta} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial x}{\partial \eta} =$$

$$= \dfrac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \xi} \dfrac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \eta} - \dfrac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \xi} \dfrac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \eta} =$$

$$= \left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}\underset{1\times 8}{} \{x_i\}_e \underset{8\times 1}{}\right)\left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}\underset{1\times 8}{} \{y_i\}_e \underset{8\times 1}{}\right) - \left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}\underset{1\times 8}{} \{y_i\}_e \underset{8\times 1}{}\right)\left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}\underset{1\times 8}{} \{x_i\}_e \underset{8\times 1}{}\right)$$

(known at any point of the domain Ω_e)

10

Gradient matrix calculation

differential operators in the coordinate system (x, y):

$$\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = [J]^{-1}_{2\times 2} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{det[J]}\dfrac{\partial y}{\partial \eta} & -\dfrac{1}{det[J]}\dfrac{\partial y}{\partial \xi} \\ -\dfrac{1}{det[J]}\dfrac{\partial x}{\partial \eta} & \dfrac{1}{det[J]}\dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} \rightarrow$$

$$\dfrac{\partial}{\partial x} = \dfrac{1}{det[J]}\dfrac{\partial y}{\partial \eta}\dfrac{\partial}{\partial \xi} - \dfrac{1}{det[J]}\dfrac{\partial y}{\partial \xi}\dfrac{\partial}{\partial \eta} \quad ; \quad \dfrac{\partial}{\partial y} = -\dfrac{1}{det[J]}\dfrac{\partial x}{\partial \eta}\dfrac{\partial}{\partial \xi} + \dfrac{1}{det[J]}\dfrac{\partial x}{\partial \xi}\dfrac{\partial}{\partial \eta}$$

gradient matrix for plane stress or plane strain conditions:

$$[R]_{3\times 2} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} = \begin{bmatrix} \left(\dfrac{1}{det[J]}\dfrac{\partial y}{\partial \eta}\dfrac{\partial}{\partial \xi} - \dfrac{1}{det[J]}\dfrac{\partial y}{\partial \xi}\dfrac{\partial}{\partial \eta}\right) & 0 \\ 0 & \left(\dfrac{1}{det[J]}\dfrac{\partial x}{\partial \xi}\dfrac{\partial}{\partial \eta} - \dfrac{1}{det[J]}\dfrac{\partial x}{\partial \eta}\dfrac{\partial}{\partial \xi}\right) \\ \left(\dfrac{1}{det[J]}\dfrac{\partial x}{\partial \xi}\dfrac{\partial}{\partial \eta} - \dfrac{1}{det[J]}\dfrac{\partial x}{\partial \eta}\dfrac{\partial}{\partial \xi}\right) & \left(\dfrac{1}{det[J]}\dfrac{\partial y}{\partial \eta}\dfrac{\partial}{\partial \xi} - \dfrac{1}{det[J]}\dfrac{\partial y}{\partial \xi}\dfrac{\partial}{\partial \eta}\right) \end{bmatrix} = [R(\xi, \eta)]_{3\times 2}$$
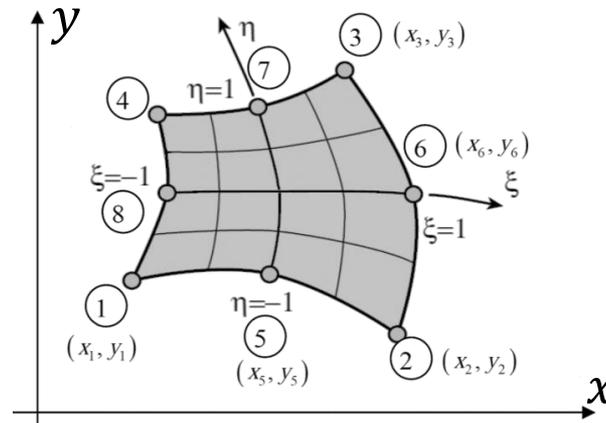
Gradient matrix for plane stress and plane strain conditions

$$[R(\xi,\eta)]_{3 \times 2} = \begin{bmatrix} \dfrac{\partial}{\partial x}(\xi,\eta) & 0 \\[3ex] 0 & \dfrac{\partial}{\partial y}(\xi,\eta) \\[3ex] \dfrac{\partial}{\partial y}(\xi,\eta) & \dfrac{\partial}{\partial x}(\xi,\eta) \end{bmatrix}$$

$$\{x_i\}_e = \begin{Bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_8 \end{Bmatrix} \quad 8 \times 1$$

$$\{y_i\}_e = \begin{Bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_8 \end{Bmatrix} \quad 8 \times 1$$

$$\frac{\partial}{\partial x}(\xi,\eta) = \frac{\left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}\{y_i\}_e\right)\dfrac{\partial}{\partial \xi} - \left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}\{y_i\}_e\right)\dfrac{\partial}{\partial \eta}}{\left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}\{x_i\}_e\right)\left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}\{y_i\}_e\right) - \left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}\{y_i\}_e\right)\left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}\{x_i\}_e\right)}$$

$$\frac{\partial}{\partial y}(\xi,\eta) = \frac{\left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}\{x_i\}_e\right)\dfrac{\partial}{\partial \eta} - \left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}\{x_i\}_e\right)\dfrac{\partial}{\partial \xi}}{\left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}\{x_i\}_e\right)\left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}\{y_i\}_e\right) - \left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}\{y_i\}_e\right)\left(\dfrac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}\{x_i\}_e\right)}$$

Strain energy of an 8-node QUAD element

strain vector for plane stress or plane strain conditions:

$$\{\varepsilon\} = \underset{3 \times 1}{[R(\xi,\eta)]}\underset{2 \times 1}{\{u\}} = \underset{3 \times 2}{[R(\xi,\eta)]}\underset{2 \times 16}{[N(\xi,\eta)]}\underset{16 \times 1}{\{q\}_e} = \underset{3 \times 16}{[B(\xi,\eta)]}\underset{16 \times 1}{\{q\}_e}$$

elastic strain energy of a finite element:

$$U_e = \frac{1}{2}\int_{\Omega_e}\underset{1 \times 3}{\lfloor\varepsilon\rfloor}\underset{3 \times 1}{\{\sigma\}}\,d\Omega_e = \frac{1}{2}\underset{1 \times 16}{\lfloor q\rfloor_e}(t_e\int_{A_e}\underset{16 \times 3}{[B(\xi,\eta)]^T}\underset{3 \times 3}{[D]}\underset{3 \times 16}{[B(\xi,\eta)]}\,dxdy)\underset{16 \times 1}{\{q\}_e} =$$

$$\underset{3 \times 1}{\{\sigma\}} = \underset{3 \times 3}{[D]}\,\underset{3 \times 1}{\{\varepsilon\}}$$

$$= \frac{1}{2}\lfloor q\rfloor_e[k]_e\{q\}_e$$

$$\underset{1 \times 3}{\lfloor\varepsilon\rfloor} = \underset{1 \times 16}{\lfloor q\rfloor_e}\underset{16 \times 3}{[B(\xi,\eta)]^T} \qquad \underset{3 \times 1}{\{\varepsilon\}} = \underset{3 \times 16}{[B(\xi,\eta)]}\underset{16 \times 1}{\{q\}_e}$$

$$(\int_{A_e} f(x,y)\,dxdy = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)det[J]\,d\xi d\eta)$$

local stiffness matrix:

$$\underset{16 \times 16}{[k]_e} = t_e\int_{-1}^{1}\int_{-1}^{1}\underset{16 \times 3}{[B(\xi,\eta)]^T}\underset{3 \times 3}{[D]}\underset{3 \times 16}{[B(\xi,\eta)]}det[J]\,d\xi d\eta \qquad \textit{(calculated numerically)}$$

13

Potential energy of loading and equivalent load vector

potential energy of loading in a finite element:

$$W_e = \int_{\Omega_e} \underset{1 \times 2}{\lfloor X \rfloor} \underset{2 \times 1}{\{u\}} \, d\Omega_e + \int_{\Gamma_{pe}} \underset{1 \times 2}{\lfloor p \rfloor} \underset{2 \times 1}{\{u\}} \, d\Gamma_{pe} =$$

$$\{u\} = \underset{2 \times 16}{[N]} \underset{16 \times 1}{\{q\}_e}$$
$$\underset{2 \times 1}{}$$

$$= \left(\int_{\Omega_e} \underset{1 \times 2}{\lfloor X \rfloor} \underset{2 \times 16}{[N]} \, d\Omega_e + \int_{\Gamma_{pe}} \underset{1 \times 2}{\lfloor p \rfloor} \underset{2 \times 16}{[N]} \, d\Gamma_{pe} \right) \underset{16 \times 1}{\{q\}_e} =$$

$$= (\underset{1 \times 16}{\lfloor F^X \rfloor_e} + \underset{1 \times 16}{\lfloor F^p \rfloor_e}) \underset{16 \times 1}{\{q\}_e} = \underset{1 \times 16}{\lfloor F \rfloor_e} \underset{16 \times 1}{\{q\}_e}$$

$$\underset{1 \times 16}{\lfloor F^X \rfloor_e} = t_e \int_{-1}^{1} \int_{-1}^{1} \underset{1 \times 2}{\lfloor X(\xi, \eta) \rfloor} \underset{2 \times 16}{[N(\xi, \eta)]} \, det[J] \, d\xi d\eta$$

$$\underset{1 \times 16}{\lfloor F^p \rfloor_e} = \int_{\Gamma_{pe}} \underset{1 \times 2}{\lfloor p \rfloor} \underset{2 \times 16}{[N]} d\Gamma_{pe}$$

$$\lfloor p \rfloor = \lfloor p_x, p_y \rfloor$$

$$\lfloor X \rfloor = \lfloor X, Y \rfloor$$

14

Equivalent load vector in the 8-node quadrilateral element

$\lfloor F \rfloor_e$
$_{16 \times 1}$
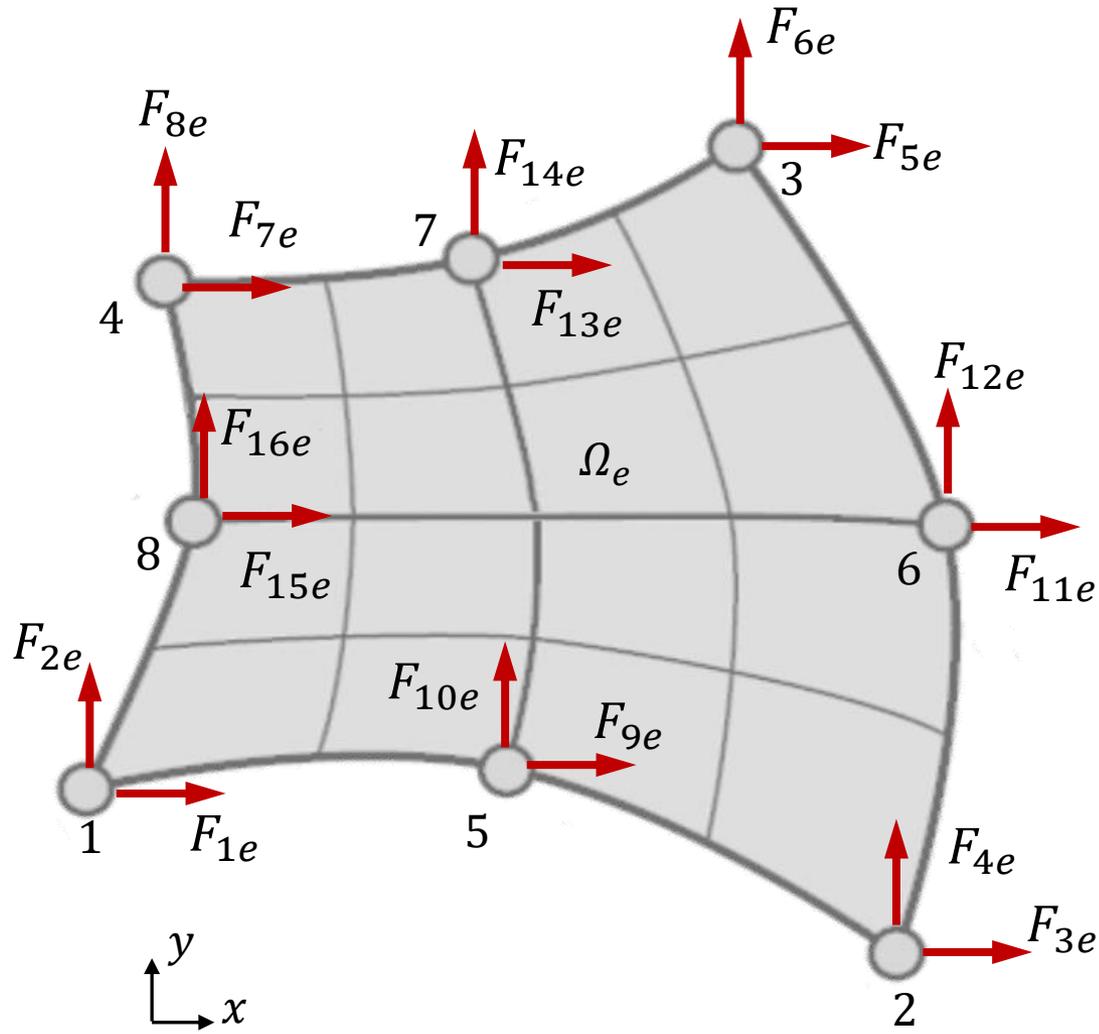
$$\{F\}_e = \begin{Bmatrix} F_1 \\ F_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ F_{16} \end{Bmatrix}_e$$

Example. Equivalent load vector due to mass forces *(gravity load)*

gravity load :

$$\lfloor F^X \rfloor_e = t_e \int_{-1}^{1} \int_{-1}^{1} \lfloor X(\xi, \eta) \rfloor [N(\xi, \eta)] \, det[J] \, d\xi d\eta$$

$$\lfloor X \rfloor_{1 \times 2}$$

$$\lfloor F^X \rfloor_e = t_e \int_{-1}^{1} \int_{-1}^{1} \lfloor 0, -\rho g \rfloor [N(\xi, \eta)] \, det[J] \, d\xi d\eta$$
$$\underset{1 \times 16}{} \qquad\qquad \underset{2 \times 16}{}$$

$$G = \rho g \Omega_e \ (N)$$

$$\lfloor F \rfloor_e = \left\lfloor 0, \frac{G}{12}, 0, \frac{G}{12}, 0, \frac{G}{12}, 0, \frac{G}{12}, 0, -\frac{G}{3}, 0, -\frac{G}{3}, 0, -\frac{G}{3}, 0, -\frac{G}{3}, \right\rfloor_e$$
$$\underset{1 \times 16}{}$$

16

Equivalent load vector due to surface load

equivalent load vector due to surface load:

$$\lfloor F^p \rfloor_e \underset{1\times 16}{} = \underset{\Gamma_{pe}}{\int} \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 16}{[N]} d\Gamma_{pe} = t_e \int_0^l \lfloor p \rfloor [N] ds$$

$$= t_e \int_0^l \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 16}{[N]} ds = t_e \int_{-1}^1 \underset{1\times 2}{\lfloor p \rfloor} \underset{2\times 16}{[N]} \frac{ds}{d\xi} d\xi$$
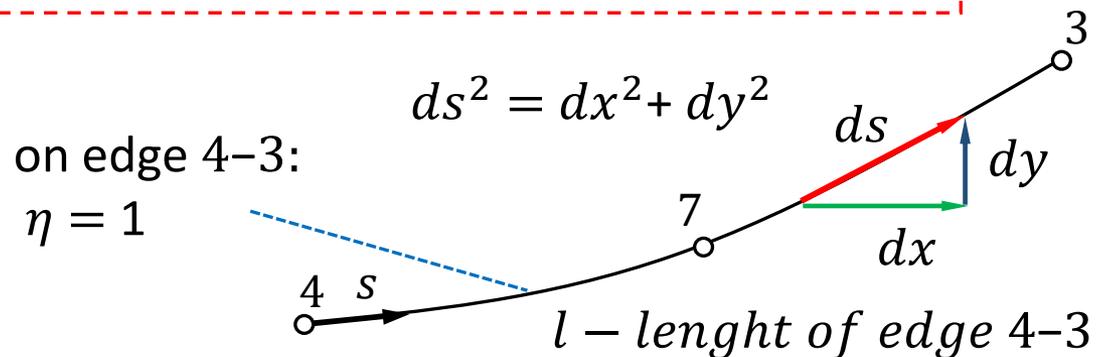
$$\frac{ds^2}{d\xi^2} = \frac{dx^2}{d\xi^2} + \frac{dy^2}{d\xi^2} \quad \rightarrow \quad \frac{ds}{d\xi} = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2}$$
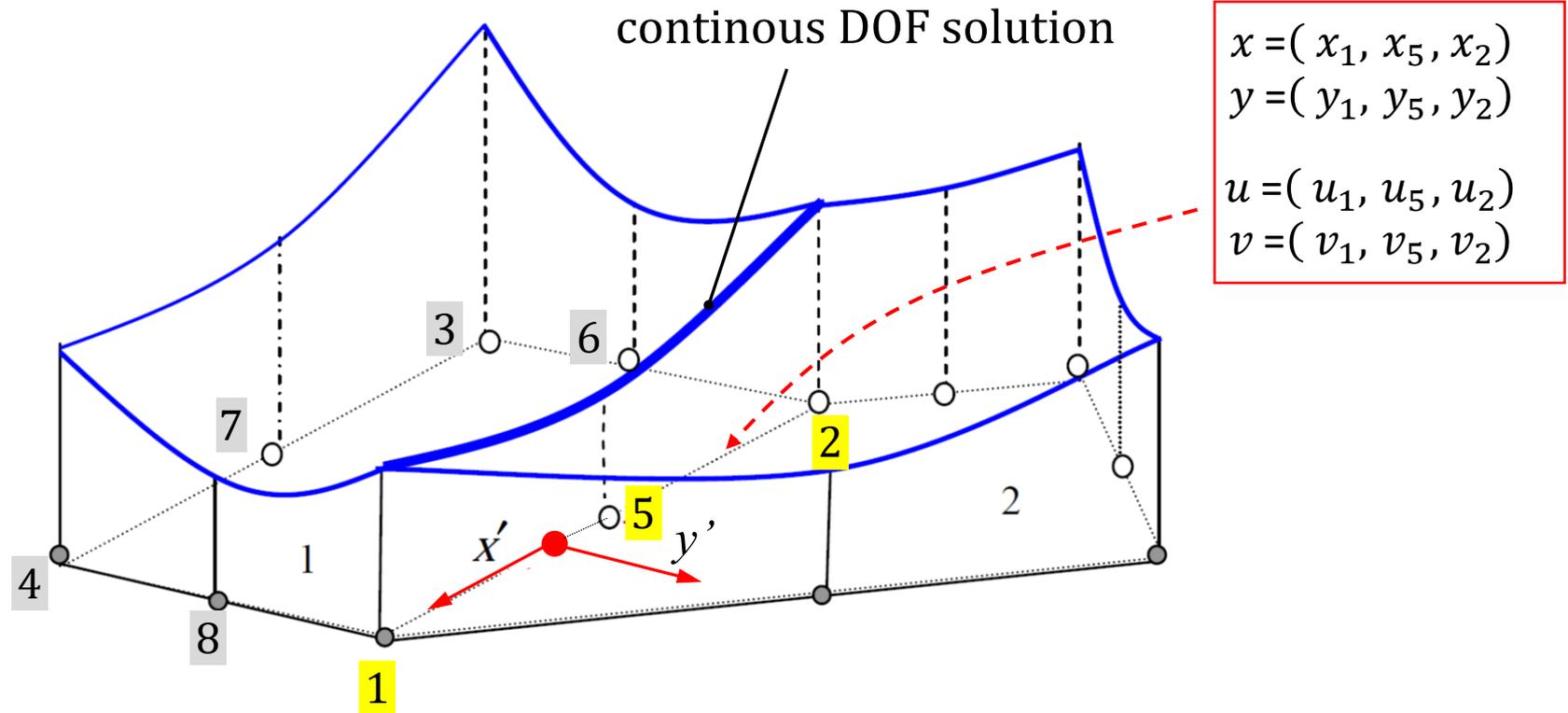
$$\lfloor F^p \rfloor_e \underset{1\times 16}{} = t_e \int_{-1}^1 \underset{2\times 16}{\lfloor p_x, p_y \rfloor [N]} \sqrt{\left(\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi} \underset{1\times 8}{} \{x_i\}_e \underset{8\times 1}{}\right)^2 + \left(\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi} \underset{1\times 8}{} \{y_i\}_e \underset{8\times 1}{}\right)^2} \, d\xi$$

(calculated numerically)

$$\lfloor p \rfloor = \lfloor p_x, p_y \rfloor$$

on edge 4–3:
$$\eta = 1$$

$$ds^2 = dx^2 + dy^2$$

$$l - \text{lenght of edge 4–3}$$

17
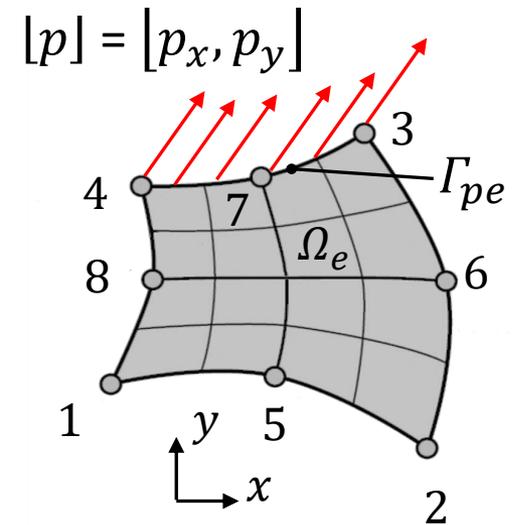
Results on the boundary between two 8-node FEs

continous DOF solution

$x =(x_1, x_5, x_2)$
$y =(y_1, y_5, y_2)$

$u =(u_1, u_5, u_2)$
$v =(v_1, v_5, v_2)$

$$\left.\frac{\partial u}{\partial x'}\right|_1 = \left.\frac{\partial u}{\partial x'}\right|_2$$

$$\left.\frac{\partial u}{\partial y'}\right|_1 \neq \left.\frac{\partial u}{\partial y'}\right|_2$$

$$\Rightarrow \quad \begin{array}{c} \left(\varepsilon_{x'}\right)_1 = \left(\varepsilon_{x'}\right)_2 \\ \left(\varepsilon_{y'}\right)_1 \neq \left(\varepsilon_{y'}\right)_2 \end{array} \Rightarrow \left(\sigma_{ij}\right)_1 \neq \left(\sigma_{ij}\right)_2$$

not continous element solution

Example. 2D model of cantilever beam *(8-node FEs)*